Chapter 5.

Rates of change.

Situation One

Research scientists are testing a new, lightweight alloy for its possible use in car, train and aeroplane bodies. In one test a vehicle made of the alloy is propelled along a straight horizontal railway in such a way that the distance, y metres, that the vehicle has travelled t seconds after it started is given by

$$y = t^3.$$

Unfortunately the vehicle starts to break up 8 seconds after it started.

The scientists knew that at that instant the vehicle was 512 metres from its starting point $(512 = 8^3)$. For those 8 seconds the average speed was given by:

$$\frac{\text{distance travelled in the 8 seconds}}{\text{time taken}} = \frac{512\text{ m}}{8\text{ sec}}$$

$$= 64\text{ m/s} \qquad (= 230{\cdot}4\text{ km/h}).$$

However the scientists want to know the speed the vehicle was travelling at the instant that it started to break up.
Try to determine the speed of the vehicle at this instant (i.e. at $t = 8$).

Situation Two

It is the year 2035 and plans are well advanced for the building of a space station on the moon. The space station will be pressurised and will act as a lunar laboratory and repair depot for space vehicles servicing the various telecommunication and surveillance satellites. The space station will obtain its power from thousands of solar tiles on the roof.
Health and Safety experts are concerned that tiles dislodged from the roof could fall on astronauts working outside the station and damage their space suits. They want tests to be carried out on earth to ensure that the space suits are strong enough to withstand the impact.
For such tests to be carried out the speed any dislodged tiles will have at the instant they reach the moon's surface is required. It is known that the tiles will fall from a height of 20 metres. Due to the moon's gravitational pull any tile will have fallen y metres, t seconds after it is dislodged, where

$$y = 0{\cdot}8t^2.$$

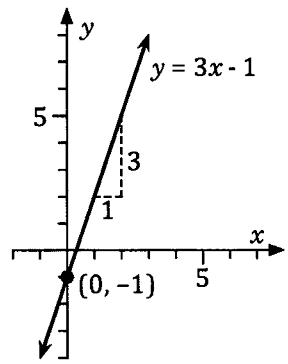
- Calculate the value of t at the instant a dislodged tile hits the surface of the moon. (Take the time the tile is dislodged as $t = 0$).
- Calculate the average speed of the tile during its fall.
- Calculate the speed of the tile at the instant it strikes the surface of the moon.

The two situations on the previous page each involved finding the speed of an object at an instant , i.e. the **instantaneous** speed. Perhaps with the first situation, having found the average speed for the motion from $t = 0$ to $t = 8$ you then considered the average speed for the motion from $t = 7$ to $t = 8$, and then perhaps you considered .... etc.

Speed is the rate at which an object changes its position with respect to time. The two situations required us to find the rate at which the distance variable (y) was changing with respect to the time variable (t), at a particular instant.

Graphically, the rate of change of one variable, y, with respect to another variable, x, is the gradient of the graph of the relationship. Therefore if either of the situations on the previous page had involved linear functions it would have been an easy matter to determine the gradient by comparison with the form $y = \text{m}x + \text{c}$, or in the situations given, $y = \text{m}t + \text{c}$. However neither $y = t^3$ nor $y = 0{\cdot}8t^2$ are of this form so, as we know, neither have straight line graphs. Therefore, if we wish to pursue this gradient idea, we first need to think about what we might mean by *the gradient of a curve.*

The gradient of a curve.

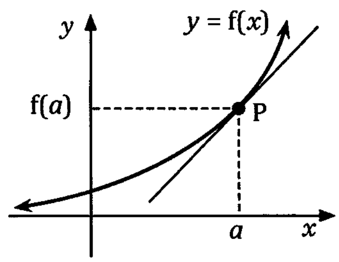


In the linear relationship shown graphed on the right, each unit increase in x sees a 3 unit increase in y.

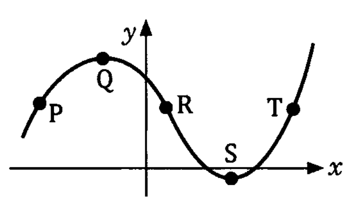
The straight line has a *constant gradient,* or *slope,* of 3.

We could say that the *rate of change of y with respect to x* is 3.

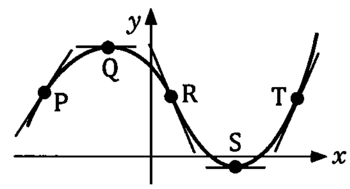
If the two variables are not linearly related the gradient, or slope, is not constant. We must then refer to the gradient at a particular point.



We define the gradient at some point P on the curve $y = \text{f}(x)$ to be the gradient of the tangent to the curve at the point P. The tangent at P is the line that "just touches" the curve at that point (except if P is a point of inflection as we will see on the next page).
If we can determine the gradient of this tangent we know the gradient of the curve at the point P and hence the rate of change of y with respect to x at $x = a$.



Consider the graph shown on the right and in particular consider the positive or negative nature of the gradient of this curve at the points P, Q, R, S and T.



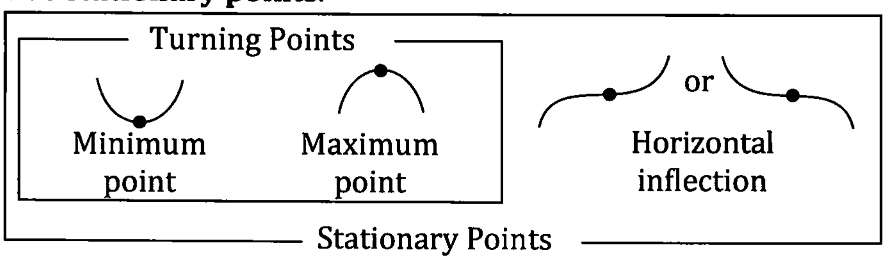
The diagram on the right now has the tangents to the curve at points P, Q, R, S and T drawn.
From this we can see that:

At points P and T the gradient is positive.
(Uphill for increasing x.)
At point R the gradient is negative.
(Downhill for increasing x.)
At points Q and S the gradient is zero. (The tangent is horizontal)

Between points P and Q we say the function is *increasing*, (a positive gradient).
Between points Q and S we say the function is *decreasing*, (a negative gradient).
Between points S and T the function is *increasing*, (a positive gradient).

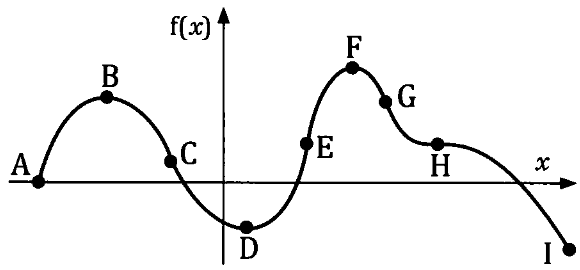
The Preliminary Work section at the beginning of this book reminded us of some other vocabulary used to describe some key features of the graphs of functions. The following dot points expand on some of this vocabulary, again referring to the graph shown above.

- In the graph the point Q is a **maximum turning point** and point S is a **minimum turning point**.
  The gradient of the curve is zero at these two points. (The tangents at these points are horizontal.)
- We can also refer to point Q as a **local maximum point** (sometimes referred to as a *relative* maximum point). There may be points on the graph that are higher than Q but *in the locality of* point Q it is the highest point.
- Looking at the section of the graph displayed the highest point overall is at the right hand end. This would be the **global maximum** for the section shown.
- Similarly point S is a **local minimum point** (or *relative* minimum point) and, for the section of graph shown, S is also the **global minimum**.
- From the extreme left of the display to point R the curve is **concave down**.
- From the point R to the extreme right of the display the curve is **concave up**.
- Point R, where the concavity changes, is a **point of inflection**. (Notice that at this point the tangent line actually cuts the curve.)
- If the graph were to continue its upward path as x increases then as x gets large positively (we say "as x **tends to infinity**" and we write "$x \to \infty$") then y also gets large positively. We write: As $x \to +\infty$ then $y \to +\infty$.
  And similarly: As $x \to -\infty$ then $y \to -\infty$.
- Local maximum and local minimum points are sometimes referred to as **turning points**. At all such points the gradient is zero.
- Maximum points, minimum points and points of horizontal inflection are sometimes referred to as **stationary points**.



Exercise 5A

1. For the function on the right, points A and I are end points, points B, D, F and H are stationary points and C, E, G and H are points of inflection.



   (a) Between which points is the function increasing, i.e. a positive gradient? (Give your answer in the form J→K, M→N, etc.)

   (b) Between which points is the function decreasing?

   (c) At which points is the gradient zero?

2. For each of the statements I → X state the letters of those graphs A → H for which the statement is true.

   I The gradient is zero at least once.

   II The gradient is always positive.

   III The gradient is always negative.

   IV The gradient is never negative.

   V The gradient is constant.

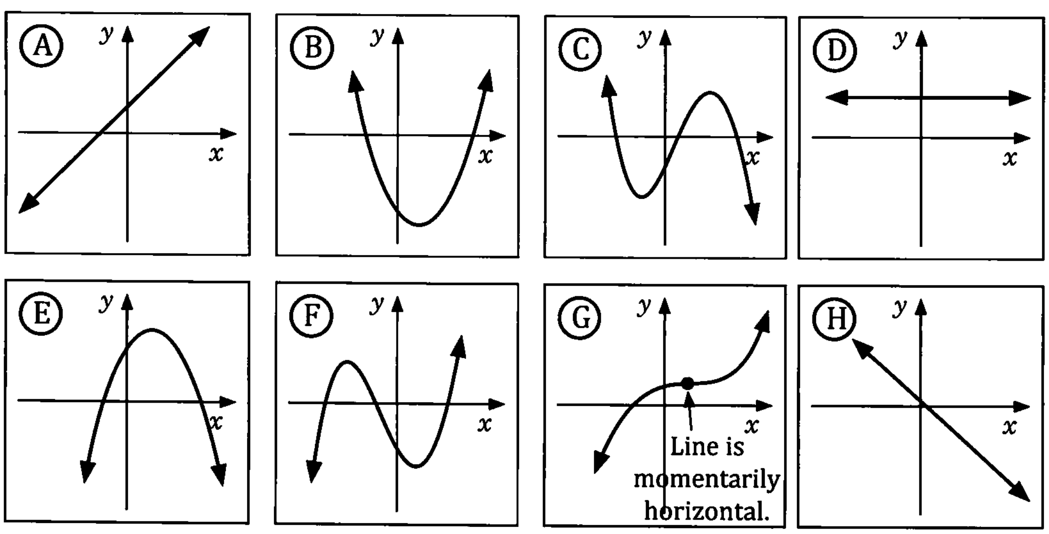
   VI The gradient is zero exactly twice.

   VII For all negative x values the gradient is positive.

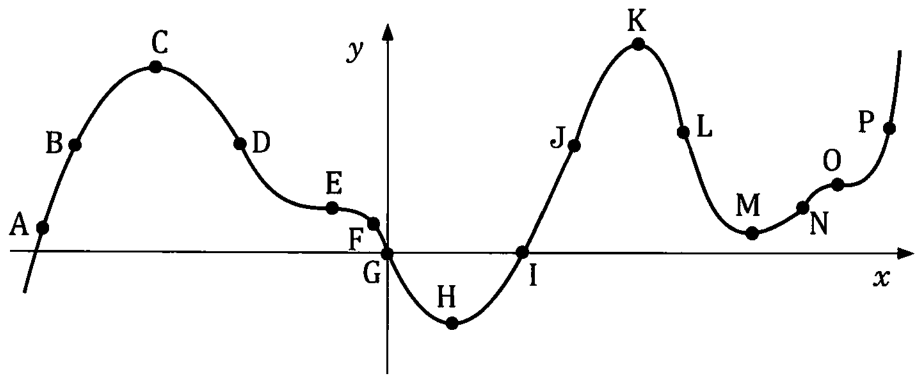
   VIII The gradient is positive as x gets very large positively. (I.e. $x \to \infty$.)

   IX The gradient is positive as x gets very large negatively. ($x \to -\infty$.)

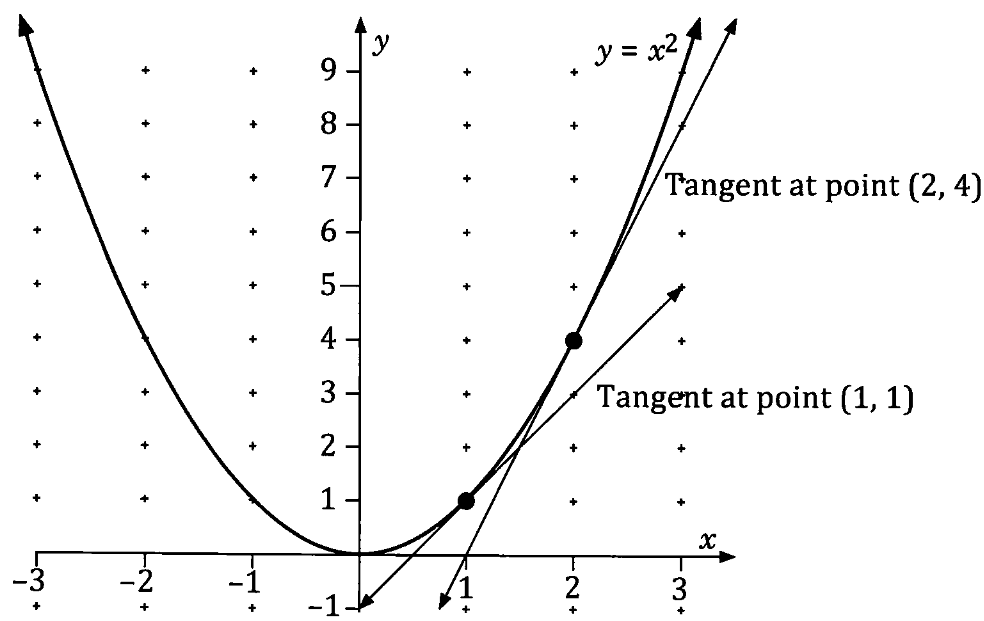
   X The gradient is negative when $x = 0$.



3. For the graph below state
   (a) which six of the points A → P are places where the gradient is zero.
   (b) which six of the points A → P are places where the gradient is positive
   (c) which four of the points A → P are places where the gradient is negative.



4. The diagram below shows the graph of $y = x^2$ with the tangents to the curve drawn at the point (1, 1) and the point (2, 4).



   (a) Use the graph to suggest the gradient of $y = x^2$ at the point (1, 1).
   (b) Use the graph to suggest the gradient of $y = x^2$ at the point (2, 4).
   (c) Use the graph to suggest the gradient of $y = x^2$ at the point (0, 0).
   (d) Suggest the gradient of $y = x^2$ at the point on the curve where $x = -1$.
   (e) Suggest the gradient of $y = x^2$ at the point on the curve where $x = -2$.
   (f) Suggest the gradient of $y = x^2 + 3$ at the point where $x = 1$.
   (g) Suggest the gradient of $y = (x - 2)^2$ at the point where $x = 3$.

5. Sketch the graph of a function that satisfies all of the conditions stated below. (You do not need to determine the equation of such a function.)
   - The function cuts the x-axis at (1, 0) and (4, 0) and nowhere else.
   - The gradient of the function is zero for $x = 2{\cdot}5$.
   - For $x < 2{\cdot}5$ the gradient is always positive.
   - For $x > 2{\cdot}5$ the gradient is always negative.

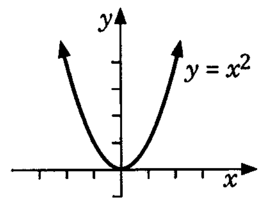
6. Sketch the graph of a function that satisfies all of the conditions stated below. (You do not need to determine the equation of such a function.)
   - The function cuts the x-axis at (0, 0) and nowhere else.
   - The gradient of the function is zero for $x = 2$.
   - For $x < 2$ and for $x > 2$ the gradient is always positive.

7. Sketch the graph of a function that satisfies all of the conditions stated below. (You do not need to determine the equation of such a function.)
   - The function cuts the x-axis at (−2, 0), (1, 0), (6, 0) and nowhere else.
   - The gradient of the function is zero for $x = -1$, $x = 3$ and $x = 5$.
   - For $x < -1$ and for $x > 5$ the gradient is always positive.
   - For $-1 < x < 3$ and $3 < x < 5$ the gradient is always negative.

Calculating the gradient at a point on a curve.

Now that we know what we mean by the gradient of a curve, how do we determine its value at various points on a curve? Well one way would be to draw the tangent to the curve at those points and estimate its gradient, as in one of the questions of the previous exercise. However, drawing the tangent accurately is difficult and deciding exactly which straight line is the tangent at a particular point involves a certain amount of guesswork. So how do we *calculate* the gradient at a particular point accurately?

To answer this question let us return to the idea mentioned after the two situations at the beginning of this chapter. It was suggested there that to determine the rate of change of the curve $y = t^3$, at the point where $t = 8$, you perhaps considered the rates of change of intervals closer and closer to $t = 8$. Let us try this approach to determine the gradient of $y = x^2$ at various points on the curve.



Consider the graph of $y = x^2$.

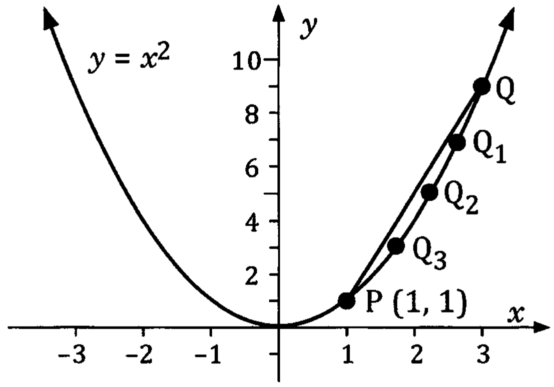
The tangent drawn through (0, 0) will be the x-axis and this has a gradient of zero.

Thus the gradient of $y = x^2$ at $x = 0$ is zero but what will be the gradient of $y = x^2$ at $x = 1, 2, 3, 4, 5, \ldots$?

For $y = x^2$

x	0	1	2	3	4	5	...
gradient	0	?	?	?	?	?	???

To determine the gradient at $x = 1$ we first determine the gradient of the **chord** PQ where P is the point (1, 1) and Q is some other point on the curve. We then move Q to positions Q_1, Q_2, Q_3, ..., each position being closer to P than the previous position, and determine the gradient of the chord in each case. As Q gets closer and closer to P then so the gradient of PQ will be a better and better approximation of the gradient of the tangent at P.



This process is shown tabulated below:

Point P	Point Q	Gradient of chord PQ
(1, 1)	(3, 9)	$\frac{9-1}{3-1} = 4$
(1, 1)	(2, 4)	$\frac{4-1}{2-1} = 3$
(1, 1)	(1·5, 2·25)	$\frac{2{\cdot}25-1}{1{\cdot}5-1} = 2{\cdot}5$
(1, 1)	(1·1, 1·21)	$\frac{1{\cdot}21-1}{1{\cdot}1-1} = 2{\cdot}1$
(1, 1)	(1·05, 1·1025)	$\frac{1{\cdot}1025-1}{1{\cdot}05-1} = 2{\cdot}05$
(1, 1)	(1·01, 1·0201)	$\frac{1{\cdot}0201-1}{1{\cdot}01-1} = 2{\cdot}01$
(1, 1)	(1·001, 1·002001)	$\frac{1{\cdot}002001-1}{1{\cdot}001-1} = 2{\cdot}001$

As Q approaches P the gradient of PQ approaches 2.

We say that the **limit** of the gradient of PQ, as Q approaches P, appears to be 2.

This suggests that the gradient of $y = x^2$ at $x = 1$ is 2.

Thus we now have:

For $y = x^2$

x	0	1	2	3	4	5	...
gradient	0	2	?	?	?	?	???

The first two questions of the next exercise involve determining more of the unknowns in this table.

Exercise 5B

1. Complete the following table to find the gradient of $y = x^2$ at the point P(2, 4).

Point P	Point Q	Gradient of chord PQ
(2, 4)	(4, 16)	$\frac{16-4}{4-2}$ = ?
(2, 4)	(3, 9)	$\frac{?-?}{?-?}$ = ?
(2, 4)	(2·5, ???)	?
(2, 4)	(2·1, ???)	?
(2, 4)	(2·01, ???)	?
(2, 4)	(2·001, ???)	?
(2, 4)	(2·0001, ???)	?

Thus the gradient of $y = x^2$ at $x = 2$ is ???.

2. Repeat the "limiting chord" process used in question 1 to determine the gradient of $y = x^2$ at (3, 9), (4, 16) and (5, 25) and hence, together with your answer from number 1, copy and complete the following table.

For $y = x^2$

x	0	1	2	3	4	5
gradient	0	2	?	?	?	?

Use your table to suggest a rule for determining the gradient of $y = x^2$ at some point (a, a^2).

3. Repeat the "limiting chord" process to determine the gradient of $y = 3x^2$ at (2, 12), (3, 27) and (4, 48) and hence copy and complete the table below.

For $y = 3x^2$

x	0	1	2	3	4	5
gradient	0	6	?	?	?	30

Use your table to suggest a rule for determining the gradient of $y = 3x^2$ at some point $(a, 3a^2)$.

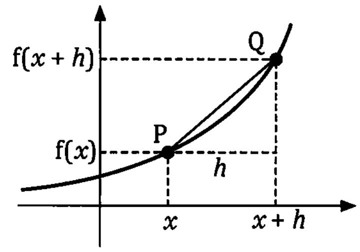
General statement of this "limiting chord" process.

Let us again consider the process we go through to find the gradient at a particular point, P, on a curve $y = f(x)$.

We choose some other point, Q, on the curve whose x-coordinate is a little more than that of point P.

Suppose P has an x-coordinate of x and Q has an x-coordinate of $(x + h)$.

The corresponding y-coordinates of P and Q will then be $f(x)$ and $f(x + h)$.



Thus the gradient of PQ = $\dfrac{f(x+h) - f(x)}{h}$

This gives us the **average rate of change** of the function from P to Q.

For example the **average rate of change** of the function $y = x^2$ from P(3, 9) to Q (4, 16) is given by $\dfrac{16-9}{4-3}$ i.e. 7

We then bring Q closer and closer to P, i.e. we allow h to tend to zero, and we determine the limiting value of the gradient of PQ.

i.e. Gradient at P = limit of $\dfrac{f(x+h) - f(x)}{h}$ as h tends to zero.

We write this as:

$$\textbf{Gradient at P } (x, f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This gives us the **instantaneous rate of change** of the function at P using an algebraic approach, rather than having to create tables as we did earlier.

For example the **instantaneous rate of change** of the function $y = x^2$ at the point P(3, 9) is given by

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h}$$

$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h}$$

$$= \lim_{h \to 0} \frac{6h + h^2}{h}$$

$$= \lim_{h \to 0} (6 + h)$$

$$= 6 \quad \text{because as } h \to 0 \text{ then so } (6 + h) \to 6$$

Does this agree with the answer you obtained numerically in question 2 of the previous exercise?

This algebraic method for determining the gradient at a point on $f(x) = x^2$ is certainly a quicker process than creating the tables that we did earlier. However, rather than using this quicker algebraic method each time we want to determine the instantaneous rate of change at some particular point on $y = x^2$, we could instead apply the technique <u>once</u> for the general point (x, x^2), obtain a formula for the gradient, and then apply this formula each time.

Consider the general point $P(x, x^2)$ lying on the function $f(x) = x^2$.
Applying the general result obtained previously:

$$\begin{aligned}\text{Gradient at } P(x, x^2) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}\\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}\\ &= \lim_{h \to 0} \frac{2xh + h^2}{h}\\ &= \lim_{h \to 0} (2x + h)\\ &= 2x \qquad \text{because as } h \to 0 \text{ then so } (2x+h) \to 2x.\end{aligned}$$

Thus for the curve $y = x^2$ the gradient formula or gradient function is $2x$.

Does this agree with your answers and suggested rule for Exercise 5B N° 2?

This process of determining the gradient formula or gradient function of a curve or function is called **DIFFERENTIATION**. (Part of the branch of mathematics known as **CALCULUS**.)

☛ If we **differentiate** $y = x^2$ with respect to the variable x we obtain the **gradient function** $2x$.

We say that $2x$ is the **derivative** of x^2.

Similarly:

☛ If we differentiate $y = t^2$ with respect to the variable t we obtain the gradient function $2t$.

☛ If we differentiate $z = y^2$ with respect to the variable y we obtain the gradient function $2y$.

☛ If we differentiate $v = z^2$ with respect to the variable z we obtain the gradient function $2z$, etc.

Exercise 5C

On the previous page the result

$$\text{Gradient at P }(x, f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

was used to determine the gradient function of $y = x^2$ as $2x$.

Use this same procedure to prove the following results.

1. The gradient function of $y = 4x^2$ is $8x$.

2. The gradient function of $y = 2x^3$ is $6x^2$.

3. The gradient function of $y = x^4$ is $4x^3$.

The results given in Exercise 5C, and that you should have found in Exercise 5B, suggest that if $y = ax^n$ then the gradient function is anx^{n-1}.

In words this general statement can be remembered as:

"multiply by the power and decrease the power by one"

This "suggested" general statement is indeed true but can we **prove** it? Well to do so we simply have to go back to the basic principle that

$$\text{Gradient at P }(x, f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

and apply it to the function $y = ax^n$.

However, before turning the page and seeing it done for you, try it yourself first.

One result that you may find useful is the *binomial expansion*, a result you were reminded of in the Preliminary Work section at the beginning of this book:

$$(p+q)^n = p^n + {}^nC_1\, p^{n-1} q^1 + {}^nC_2\, p^{n-2} q^2 + {}^nC_3\, p^{n-3} q^3 + \ldots + {}^nC_n\, p^0 q^n$$

An alternative approach would be to use another result that was mentioned in the Preliminary work:

$$p^n - q^n = (p-q)(p^{n-1} + p^{n-2}q + p^{n-3}q^2 + p^{n-4}q^3 + \ \ldots \ pq^{n-2} + q^{n-1})$$

Have a go!

Consider some general point $P(x, ax^n)$ on $f(x) = ax^n$.

The gradient at $P(x, ax^n)$ $= \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \dfrac{a(x+h)^n - ax^n}{h}$ ← ①

$= \lim_{h \to 0} \dfrac{a[(x+h)^n - x^n]}{h}$

$= \lim_{h \to 0} \dfrac{a(x^n + {}^nC_1 x^{n-1} h + {}^nC_2 x^{n-2} h^2 + \ldots + h^n - x^n)}{h}$

$= \lim_{h \to 0} \dfrac{a({}^nC_1 x^{n-1} h + {}^nC_2 x^{n-2} h^2 + \ldots + h^n)}{h}$

$= \lim_{h \to 0} (a\,{}^nC_1 x^{n-1} + a\,{}^nC_2 x^{n-2} h + a\,{}^nC_3 x^{n-3} h^2 \ldots ah^{n-1})$

$= a\,{}^nC_1 x^{n-1}$

$= a\,n\,x^{n-1}$

The reader is left to show that the same result can be arrived by applying the rule given at the bottom of the previous page to equation ① above

Notation.

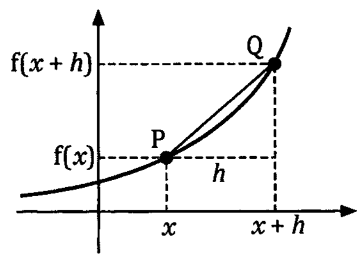
In the expression

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

"h" is a small **increment** in the variable x and

$$[f(x+h) - f(x)]$$

is the corresponding small **increment** in the variable y.



Denoting this small increment in x as δx, where "δ" is a Greek letter pronounced "delta", and the small increment in y as δy, we have:

Gradient function $= \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

which we write as $\dfrac{dy}{dx}$ (pronounced "dee y by dee x").

Thus if $y = x^2$ then $\frac{dy}{dx}$, the gradient function, is $2x$.

If $y = x^3$ then $\frac{dy}{dx}$, the gradient function, is $3x^2$.

If $y = x^4$ then $\frac{dy}{dx}$, the gradient function, is $4x^3$.

> If $y = ax^n$ then $\frac{dy}{dx}$, the gradient function, is anx^{n-1}.

This can also be written as

$$\frac{d}{dx}(y) = \frac{d}{dx}(ax^n)$$
$$= anx^{n-1}$$

We say "dee by dee x" of ax^n is anx^{n-1}. I.e. the derivative of ax^n is anx^{n-1}.

Example 1

Determine the gradient function for each of the following.

(a) $y = 3x^2$ (b) $y = 7x^3$ (c) $y = 2x^5$ (d) $y = 3x$ (e) $y = 7$

(a) If $y = 3x^2$

then $\frac{dy}{dx} = 3(2)x^{2-1}$

$= 6x$

(b) If $y = 7x^3$

then $\frac{dy}{dx} = 7(3)x^{3-1}$

$= 21x^2$

(c) If $y = 2x^5$

then $\frac{dy}{dx} = 2(5)x^{5-1}$

$= 10x^4$

(d) If $y = 3x$ (i.e. $3x^1$)

then $\frac{dy}{dx} = 3(1)x^{1-1}$

$= 3$

(as expected because $y = 3x$ is a straight line with gradient 3.)

(e) If $y = 7$ (i.e. $7x^0$)

then $\frac{dy}{dx} = 7(0)x^{0-1}$

$= 0$ (as we would expect because $y = 7$ is a horizontal line.)

The answers for the derivatives in the previous example can also be obtained from some calculators.

$\frac{d}{dx}(3x^2)$ $\quad 6\cdot x$

$\frac{d}{dx}(7x^3)$ $\quad 21\cdot x^2$

$\frac{d}{dx}(2x^5)$ $\quad 10\cdot x^4$

$\frac{d}{dx}(3x)$ $\quad 3$

$\frac{d}{dx}(7)$ $\quad 0$

Example 2

Determine the gradient of the curve $y = 3x^4$ at the point (2, 48).

If $\quad y = 3x^4$ then $\frac{dy}{dx} = 12x^3$

At (2, 48) $\quad x = 2$. Thus $\frac{dy}{dx} = 12\,(2)^3$

$= 96.$

The gradient of $y = 3x^4$ at (2, 48) is 96.

$\frac{d}{dx}(3\cdot x^4) \mid x = 2$ $\quad 96$

Note: For the moment we are differentiating functions of the form $y = ax^n$ for n a non-negative integer. Later in this chapter we will consider more general **polynomial functions** which, as the reader should know, are of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \;.....\; + a_2 x^2 + a_1 x + a_0$$

where n is a non-negative integer and $a_n, a_{n-1}, a_{n-2}, ...$ are all numbers, called the **coefficients** of x^n, x^{n-1}, x^{n-2} etc.

The highest power of x is the **order** of the polynomial.

Thus linear functions, $\quad y = mx + c$, $\quad$ are polynomials of order 1.

quadratic functions, $\quad y = ax^2 + bx + c$, $\quad$ are polynomials of order 2.

cubic functions, $\quad y = ax^3 + bx^2 + cx + d$, $\quad$ are polynomials of order 3. Etc.

Get to know the capability of your calculator with regard to finding the derivative of a function and of finding the value of the derivative for a specific x-value. However make sure that if the course requires it you can also determine derivatives, and gradients at a point, yourself, without access to a calculator.

Example 3

Find the coordinates of any points on the curve $y = x^3$ where the gradient is 12.

If $y = x^3$ then $\frac{dy}{dx} = 3x^2$

Thus we require points for which $3x^2 = 12$

i.e. $x^2 = 4$

giving $x = 2$ or $x = -2$

If $x = 2$, $y = 2^3 = 8$ and if $x = -2$ $y = (-2)^3 = -8$

Thus $y = x^3$ has a gradient of 12 at (2, 8) and (−2, −8).

Note

- If $y = f(x)$ then the derivative of y with respect to x can be written as

  $$\frac{dy}{dx}, \quad \frac{df}{dx} \quad \text{or} \quad \frac{d}{dx}f(x).$$

  (This last version is pronounced: *"Dee by dee x of eff of x".*)

- A shorthand notation using a "dash" is sometimes used for differentiation with respect to x.

  Thus if $y = f(x)$ we can write $\frac{dy}{dx}$ as $f'(x)$ or simply y' or f'.

Example 4

Determine $f'(x)$ for (a) $f(x) = 7x^5$, (b) $f(x) = 20$, (c) $f(x) = 6x^9$.

(a) If $f(x) = 7x^5$ then $f'(x) = 35x^4$

(b) If $f(x) = 20$ then $f'(x) = 0$

(c) If $f(x) = 6x^9$ then $f'(x) = 54x^8$

Finding the equation of a tangent at a point on $y = ax^n$.

Example 5

Find the equation of the tangent to $y = 0{\cdot}5x^3$ at the point (2, 4).

Either algebraically: or by calculator:

If $y = 0{\cdot}5x^3$ then $\frac{dy}{dx} = 1{\cdot}5x^2$

Thus at (2, 4) $\frac{dy}{dx} = 1{\cdot}5(2)^2$

$= 6$

$\frac{d}{dx}(0{\cdot}5x^3)\big|\, x = 2$

6

We first determine that the gradient of the curve $y = 0{\cdot}5x^3$, at the point (2, 4), is 6.
Thus the gradient of the tangent to $y = 0{\cdot}5x^3$, at the point (2, 4), is also 6.
The tangent is a straight line and has an equation of the form $y = 6x + c$
But (2, 4) lies on this tangent $\therefore$ $4 = 6(2) + c$
giving $c = -8$
The tangent to $y = 0{\cdot}5x^3$ at the point (2, 4) has equation $y = 6x - 8$.

Some calculators and internet programs are able to determine the equation of a tangent at a point on a curve directly, given the appropriate instructions. Whilst you are encouraged to explore this capability of such programs make sure you can carry out the process shown in the above example yourself.

Though we have been concentrating on finding the gradients at points on various curves it is important to remember that the gradient tells us the rate at which one variable is changing with respect to another. Rates of change are important in everyday life. Differentiation can be used to find:

- the rate at which a vehicle is changing its position with respect to time, i.e. the vehicle's speed.
- the rate of change in the population of a country.
- the rate of change in the number of people suffering a disease.
- the rate of change in the value of one currency with respect to another.
- the rate of change in the total profit we get from a particular item with respect to the unit cost of that item. Etc.

In the remainder of this chapter we will concentrate on improving our ability to differentiate various functions. In the next chapter we will apply these skills to some real life rate of change situations.

Exercise 5D

Determine the gradient function $\frac{dy}{dx}$ for each of the following.

1. $y = x^2$	2. $y = x^3$	3. $y = x$	4. $y = x^4$
5. $y = 3$	6. $y = 6x^2$	7. $y = 6x^4$	8. $y = 7x$
9. $y = 16x$	10. $y = 2x^7$	11. $y = 7x^2$	12. $y = 9x$
13. $y = \frac{x^2}{10}$	14. $y = \frac{2x^6}{3}$	15. $y = \frac{3x^6}{2}$	16. $y = \frac{2x^7}{7}$

Differentiate each of the following with respect to x.

17. $4x^2$	18. $5x^4$	19. $8x^3$	20. 9
21. x^7	22. $4x^6$	23. $9x^2$	24. $5x$

Determine $f'(x)$ for each of the following.

25. $f(x) = 5$	26. $f(x) = 6x^3$	27. $f(x) = 8x^4$	28. $f(x) = 3x^5$
29. $f(x) = x^6$	30. $f(x) = 6x^7$	31. $f(x) = 4x^4$	32. $f(x) = 10x$

Determine the gradient of each of the following at the given point.

33. $y = 2x^2$ at the point $(3, 18)$	34. $y = 4x^3$ at the point $(1, 4)$
35. $y = 4x^3$ at the point $(-1, -4)$	36. $y = x^5$ at the point $(2, 32)$
37. $y = 7x$ at the point $(2, 14)$	38. $y = 5x^2$ at the point $(-2, 20)$
39. $y = 0{\cdot}25x^2$ at the point $(4, 4)$	40. $y = \frac{x^2}{5}$ at the point $(2, 0{\cdot}8)$

Find the coordinates of the point(s) on the following curves where the gradient is as stated

41. $y = x^4$. Gradient 4.
42. $y = x^3$. Gradient 3.
43. $y = 3x^2$. Gradient 9.
44. $y = 2x^3$. Gradient 1·5.
45. $y = x^6$. Gradient 6.
46. $y = x^6$. Gradient −6.

Find the equation of the tangent to the following curves at the indicated point.

47. $y = 2x^3$ at the point $(1, 2)$
48. $y = 3x^2$ at the point $(-1, 3)$
49. $y = 5x^2$ at the point $(2, 20)$
50. $y = 5x^2$ at the point $(-2, 20)$
51. $y = \frac{x^4}{2}$ at the point $(2, 8)$
52. $y = \frac{x^3}{6}$ at the point $(6, 36)$

53. If $f(x) = 3x^3$ find (a) $f(2)$ (b) $f(-1)$ (c) $f'(x)$ (d) $f'(2)$

54. If $f(x) = 1{\cdot}5x^2$ find (a) $f(2)$ (b) $f(4)$ (c) $f'(x)$ (d) $f'(2)$

55. For $y = 2x^3$ determine:
    (a) By how much y changes when x changes from $x = 2$ to $x = 5$.
    (b) The average rate of change in y, per unit change in x, when x changes from $x = 2$ to $x = 5$.
    (c) The instantaneous rate of change of y, with respect to x, when $x = 2$.
    (d) The instantaneous rate of change of y, with respect to x, when $x = 5$.

56. The straight line $y = 8x + 16$ cuts the curve $y = 8x^2$ at two points. Find the coordinates of each point and the gradient of the curve at each one.

57. The straight line $y = 4x$ cuts the curve $y = x^3$ at three points. Find the coordinates of each point and the gradient of the curve at each one.

58. The tangent to the curve $y = ax^4$ at the point $(3, b)$ has a gradient of 2. Find the values of a and b.

59. The tangent to the curve $y = ax^3$ at the point $(-1, b)$ is perpendicular to the line $y = 2x + 3$. Find the values of a and b.
    (As mentioned in the *Preliminary work*: If two lines are perpendicular the product of their gradients is −1.)

Differentiating $f(x) \pm g(x)$.

We know that if $y = 3x$ then $\frac{dy}{dx} = 3$ and if $y = x^2$ then $\frac{dy}{dx} = 2x$.

It then seems reasonable to suggest that if $y = 3x + x^2$ then $\frac{dy}{dx} = 3 + 2x$.

To check whether this seemingly reasonable suggestion is true we differentiate $3x + x^2$ from first principles, i.e. by determining $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Let $\quad f(x) = 3x + x^2$

then $\quad f(x+h) = 3(x+h) + (x+h)^2$.

Thus
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h) + (x+h)^2] - [3x + x^2]}{h}$$
$$= \lim_{h \to 0} \frac{3x + 3h + x^2 + 2xh + h^2 - 3x - x^2}{h}$$
$$= \lim_{h \to 0} (3 + 2x + h)$$
$$= 3 + 2x$$

Though the above example only considers the particular function $(3x + x^2)$ it is in fact true that if $\quad y = f(x) \pm g(x)$

then $\quad \frac{dy}{dx} = f'(x) \pm g'(x)$

Note:

- (For interest.)
  If some operation, which we will call Z, applied to the functions f and g is
  such that $\quad Z(f + g) = Z(f) + Z(g)$
  and $\quad Z(k \times f) = k \times Z(f) \quad$ for k a constant
  then K is said to be a *linear* operator.
  Hence the facts that $\quad \frac{d}{dx}(ky) = k\frac{dy}{dx} \quad$ for k a constant
  and $\quad \frac{d}{dx}(y_1 + y_2) = \frac{dy_1}{dx} + \frac{dy_2}{dx}$
  confirm what may be referred to as the "linearity property" of derivatives.

- (For interest.)
  The use of δx and δy to represent small increments in x and y respectively, and the use of the term $\frac{dy}{dx}$, is referred to as "Leibniz notation" in honour of the German Mathematician Gottfried Leibniz (1646 – 1716).

Example 6

Find the gradient of $y = x^2 - 3x$ at the point $(5, 10)$.

By calculator (typical display below):

$$\frac{d}{dx}(x^2 - 3\cdot x)\Big|\, x = 5$$

7

Algebraically:

If $\quad y = x^2 - 3x$

then $\quad \frac{dy}{dx} = 2x - 3$

Therefore, at the point $(5, 10)$

$$\frac{dy}{dx} = 2(5) - 3$$
$$= 7$$

The gradient of $y = x^2 - 3x$ at $(5, 10)$ is 7.

Exercise 5E

Find the gradient function $\frac{dy}{dx}$ for each of the following.

1. $y = x^2 + 3x$
2. $y = x^3 - 4x + 7$
3. $y = 6x^2 - 7x^3 + 4$
4. $y = 3x^4 + 2x^3 - 5x$
5. $y = 6 + 7x + x^2$
6. $y = 6x^2 - 3x$
7. $y = 4x^2 + 7x - 1$
8. $y = 5x^3 - 4x^2 + 8$
9. $y = 5x^4 - 3x + 11$
10. $y = 2x^2 + 7x + 1$
11. $y = 5 - 3x^2 + 7x$
12. $y = 1 + x + x^2 + x^3 + x^4$
13. $y = 5 - 4x + 3x^2 - 2x^3 + x^4$

14. Find the gradient of $y = x^3 - 3x^2$ at the point $(1, -2)$.
15. Find the gradient of $y = 17 + 2x^3$ at the point $(-2, 1)$.
16. Find the gradient of $y = x^3 - x^2 - 8$ at the point $(3, 10)$.
17. Find the gradient of $y = 1 + 3x - 2x^3 + x^4$ at the point $(2, 7)$.

18. Find the equation of the tangent to $y = x^2 + 3x$ at the point $(2, 10)$.
19. Find the equation of the tangent to $y = 2x^2 - 7x$ at the point $(5, 15)$.
20. Find the equation of the tangent to $y = x^3 - 5x^2 + 14$ at the point $(4, -2)$.
21. Find the equation of the tangent to $y = 5x^4 - 4x^5$ at the point $(1, 1)$.

22. Find the coordinates of any point on the curve $y = x^3 + 6x^2 - 10x + 1$ where the gradient is 5.

23. The curve $y = x^2 - 2x - 15$ cuts the x-axis in two places. Find the coordinates of each of these points and determine the gradient of the curve at each one.

24. Find the coordinates of any point on the curve $y = x^2 - 7x$ where the gradient is the same as that of $3y = 9x - 1$.

25. Find the coordinates of any point on the curve $y = x^3 + 3x^2 - 7x - 1$ where the gradient is the same as that of $y = 2x + 3$.

Differentiating more general power functions.

Functions of the form $y = ax^n$ are called **power functions** and in this chapter we have considered such functions for non-negative integer values of n. We have differentiated such functions using the fact that if

$$y = ax^n \quad \text{then} \quad \frac{dy}{dx} = anx^{n-1}.$$

Polynomial functions, which are linear combinations of such power functions, could then be differentiated using the fact that if

$$y = f(x) \pm g(x) \quad \text{then} \quad \frac{dy}{dx} = f'(x) \pm g'(x).$$

Now let us remove the restriction that n must be a non-negative integer and consider more general power functions $y = ax^n$, for example $y = \sqrt{x}$ and $y = \frac{1}{x}$.

Though not proved here, it is the case that for negative and fractional values of n, the same rule applies, i.e, if $\mathbf{y = ax^n}$ **then** $\mathbf{\frac{dy}{dx} = anx^{n-1}}$.

Thus if $y = \frac{1}{x}$ i.e. $y = x^{-1}$ then $\frac{dy}{dx} = -1x^{-2} = -\frac{1}{x^2}$

and if $y = \sqrt{x}$ i.e. $y = x^{\frac{1}{2}}$ then $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx}\left(\frac{1}{x}\right)$	$\frac{-1}{x^2}$
$\frac{d}{dx}(\sqrt{x})$	$\frac{1}{2\cdot\sqrt{x}}$

It then follows that:

If $y = 3x^2 - 2x + 7 + \sqrt[3]{x^2} - \frac{5}{x^2}$ i.e. $y = 3x^2 - 2x + 7 + x^{\frac{2}{3}} - 5x^{-2}$

$$\frac{dy}{dx} = 6x - 2 + \frac{2}{3}x^{-\frac{1}{3}} + 10x^{-3}$$

$$= 6x - 2 + \frac{2}{3\sqrt[3]{x}} + \frac{10}{x^3}$$

Example 7

Find the gradient of $y = x^2 + \frac{16}{x}$ at the point (4, 20).

By calculator (typical display below):

$\frac{d}{dx}\left(x^2 + \frac{16}{x}\right)\Big|x = 4$

7

Algebraically:

If $y = x^2 + \frac{16}{x}$

$= x^2 + 16x^{-1}$

then $\frac{dy}{dx} = 2x - 16x^{-2}$

Therefore, at the point (4, 20)

$\frac{dy}{dx} = 2(4) - 16(4)^{-2}$

$= 7$

The gradient of $y = x^2 + \frac{16}{x}$ at (4, 20) is 7.

Example 8

Find the equation of the tangent to $y = 12\sqrt{x}$ at the point (4, 24).

Either algebraically: or by calculator:

If $y = 12x^{\frac{1}{2}}$ then $\frac{dy}{dx} = 6x^{-\frac{1}{2}}$

Thus at (4, 24) $\frac{dy}{dx} = 6(4)^{-\frac{1}{2}}$

$= \frac{6}{\sqrt{4}}$

$= 3$

$\frac{d}{dx}(12\sqrt{x})\Big|x = 4$

3

We determine that the gradient of the curve $y = 12\sqrt{x}$, at the point (4, 24), is 3.

Thus the gradient of the tangent to $y = 12\sqrt{x}$, at the point (4, 24), is also 3.

The tangent, being a straight line, will have equation of the form $y = 3x + c$

But (4, 24) lies on this tangent $\therefore\ 24 = 3(4) + c$

giving $c = 12$

The required tangent has equation $y = 3x + 12$.

Exercise 5F

(Whilst you are encouraged to use your calculator to obtain expressions for the derivative, and to determine its value at particular points on a curve, it is suggested that you do most of the following questions algebraically to ensure that you can follow the basic processes without a calculator.)

Determine the gradient function $\frac{dy}{dx}$ for each of the following.

1. $y = \sqrt{x}$
2. $y = \frac{1}{x}$
3. $y = \frac{3}{x}$
4. $y = 6x^{\frac{1}{2}}$
5. $y = 6x^{\frac{1}{3}}$
6. $y = \sqrt{x^3}$
7. $y = 2\sqrt[3]{x}$
8. $y = \frac{1}{x^3}$
9. $y = \frac{1}{x^4}$
10. $y = \frac{2}{x^3}$
11. $y = \frac{5}{x^4}$
12. $y = x^2 + \sqrt{x}$
13. $y = 3x^2 - 4\sqrt{x}$
14. $y = x + \frac{1}{x}$
15. $y = x^2 - \frac{1}{x^2}$
16. $y = \sqrt{x} + \frac{3}{x}$
17. $y = x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2}$

Determine f'(x) for each of the following.

18. $f(x) = \frac{2}{x}$
19. $f(x) = \frac{3}{\sqrt{x}}$
20. $f(x) = \frac{6}{\sqrt[3]{x}}$
21. $f(x) = \frac{1}{\sqrt[3]{x}}$

22. Find the gradient of $y = \frac{4}{x} - x^2$ at the point (2, −2).

23. Find the gradient of $y = \frac{1}{x^2}$ at the point $(-2, \frac{1}{4})$.

24. Find the gradient of $y = 1 - \frac{1}{x}$ at the point (4, 0·75).

25. Find the gradient of $y = 3x^3 - \frac{2}{x}$ at the point (1, 1).

26. Find the gradient of $y = \sqrt[3]{(x^4)}$ at the point $(8, 16)$.

27. Find the gradient of $y = 6\sqrt[3]{x} + \frac{2}{x^3}$ at the point $(1, 8)$.

28. Find the gradient of $y = \frac{2}{x} + x^2 + \frac{16}{x^2}$ at the point $(2, 9)$.

29. Find the coordinates of the point(s) on the curve $y = \frac{1}{x}$ where the gradient is equal to $-\frac{1}{4}$.

30. Find the coordinates of the point(s) on the curve $y = \sqrt{x}$ where the gradient is equal to 1.

31 Find the coordinates of any point on the curve $y = x^2 - 108\sqrt{x}$ where the gradient is zero.

32. Find the equation of the tangent to the curve $y = \sqrt{x}$ at the point $(4, 2)$.

33. Find the equation of the tangent to the curve $y = \frac{1}{x}$ at the point $(1, 1)$.

34. Find the equation of the tangent to the curve $y = \frac{1}{x^2}$ at the point $(2, 0{\cdot}25)$.

35. Find the coordinates of any point on the curve $y = 2x - \frac{1}{x}$ where the gradient is the same as that of $16y = 41x + 6$.

36. Challenge.
Use the first principles definition

$$\text{Gradient at P } (x, \mathrm{f}(x)) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

to show that if $y = \frac{1}{x}$ then $\frac{dy}{dx} = -\frac{1}{x^2}$

and if $y = \sqrt{x}$ then $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.

Miscellaneous Exercise Five.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Find the value of n in each of the following.

(a) $5 \times 5 \times 5 \times 5 = 5^n$	(b) $2^4 = n$	(c) $2^n = 8$
(d) $6^3 \times 6^4 = 6^n$	(e) $2^6 \times 8 = 2^n$	(f) $3^2 \times 3^n = 3^6$
(g) $100 \times 10^n = 10^6$	(h) $16 \times 8 = 2^n$	(i) $4 \times 16 = 4^n$
(j) $8^5 \div 8^n = 8^2$	(k) $15^n = 1$	(l) $3^2 \times 3^n \times 3 = 3^7$
(m) $5^9 \div 5^3 \times 5^n = 5^8$	(n) $5^9 \div (5^3 \times 5^n) = 5^2$	(o) $8 \times 8 \times 8 = 2^n$

2. (a) Find the average rate of change of the function $y = x^2$ from the point P(4, 16) to the point Q (5, 25).
   (b) Find the instantaneous rate of change of the function $y = x^2$ at the point with coordinates (8, 64).

3. (a) Find the average rate of change of the function $y = x^3$ from the point on the curve where $x = 1$ to the point on the curve where $x = 3$.
   (b) Find the instantaneous rate of change of the function $y = -2x^3$ at the point on the curve with coordinates (−2, 16).

4. If we consider our "ancestors" to be our parents, grandparents, great grandparents, great great grandparents etc. then going back one generation we each have two ancestors in that generation, going back two generations we each have four ancestors in that generation, etc. (Assuming no repetition of ancestors).
   How many ancestors has a person got (a) in the tenth generation back,
   (b) in the thirtieth generation back?
   How many ancestors has a person got altogether if we sum the ancestors from
   (c) the first generation back to the tenth generation back,
   (d) the first generation back to the thirtieth generation back?

5. Differentiate each of the following with respect to x.
   (a) $5 - x^3$ (b) $5x^2 - 6\sqrt{x}$ (c) $5x^2 + 6 + \frac{1}{x^2}$

6. Find the gradient of $y = \frac{1}{x}$ at the point (0·5, 2).

7. ☛ For each of the tables shown below determine whether the relationship that exists between x and y is:

linear, (rule can be written in the form $y = mx + c$),
quadratic, (rule can be written in the form $y = ax^2 + bx + c$),
cubic, (rule can be written in the form $y = ax^3 + bx^2 + cx + d$),
exponential, (rule can be written in the form $y = a \times b^x$),
reciprocal, (rule can be written in the form $y = \frac{k}{x}$),

or none of the above five types.

And:

☛ For those that are one of the above five types determine the algebraic rule for the relationship in the form "y = ???".
("undef " indicates that the function is undefined for that value of x.)

(a)

x	−4	−3	−2	−1	0	1	2	3	4
y	1·5	2	3	6	undef	−6	−3	−2	−1·5

(b)

x	−4	−3	−2	−1	0	1	2	3	4
y	17	10	5	2	1	2	5	10	17

(c)

x	−4	−3	−2	−1	0	1	2	3	4
y	−7	−4	−1	2	5	8	11	14	17

(d)

x	−4	−3	−2	−1	0	1	2	3	4
y	0·0016	0·008	0·04	0·2	1	5	25	125	625

(e)

x	−4	−3	−2	−1	0	1	2	3	4
y	12	6	2	0	0	2	6	12	20

Hint: To obtain the rule consider the y values as:
−4 × −3 −3 × −2 −2 × −1 −1 × 0 0 × 1 1 × 2 2 × 3 3 × 4 4 × 5

(f)

x	−4	−3	−2	−1	0	1	2	3	4
y	0·0001	0·001	0·01	0·1	1	10	100	1000	10000

(g)

x	−4	−3	−2	−1	0	1	2	3	4
y	0·25	0·5	1	2	4	8	16	32	64

(h)

x	−4	−3	−2	−1	0	1	2	3	4
y	6	8	12	24	undef	−24	−12	−8	−6

(i)

x	−4	−3	−2	−1	0	1	2	3	4
y	−56	0	20	16	0	−16	−20	0	56

8. A triangle has its three angles in Arithmetic Progression. If the smallest angle is 10° find the size of the other two angles.

9. A particular sequence is geometric with a common ratio of 5 and a fourth term equal to 100. Define the sequence by stating the first term, T_1, and giving T_{n+1} in terms of T_n.

10. Given that $a = 2 \times 10^7$ and $b = 4 \times 10^4$ evaluate each of the following, without the assistance of a calculator, giving your answers in standard form (scientific notation).

(a)	$a \times b$	(b)	$b \times a$	(c)	a^3
(d)	b^2	(e)	$b \div a$	(f)	$a \div b$

11. For each of the following sequences:

Sequence 1:	$T_1 = 5$	and	$T_{n+1} = 3T_n + 2$
Sequence 2:	$T_1 = 0{\cdot}125$	and	$T_{n+1} = T_n \times 2$
Sequence 3:	$T_1 = -5$	and	$T_{n+1} = T_n + 10$

(a) State the first five terms.
(b) State whether the sequence is arithmetic, geometric or neither of these.
(c) State the sum of the first five terms.
(d) State the eighteenth term. (Use a calculator or spread sheet.)
(e) State the sum of the first 18 terms. (Use a calculator or spread sheet.)

12. Find the equation of the tangent to the curve $y = 2x^3 - x + 3$
(a) at the point (1, 4),
(b) at any points on the curve where the gradient is 23.

13. What do each of the following displays tell us about the rate of change of

$$f(x) = x^3 + 3x^2 + 4$$

(a)

Define $f(x) = x^3 + 3x^2 + 4$

$$\frac{f(6) - f(1)}{5}$$

64

(b)

Define $f(x) = x^3 + 3x^2 + 4$

$$\lim_{h \to 0} \left(\frac{f(5+h) - f(5)}{h} \right)$$

105

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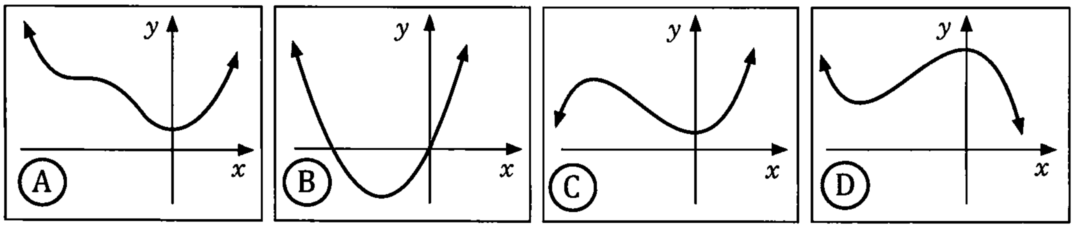
14. One of the graphs A to D shown below has $\frac{dy}{dx} = x(x+3)$. Which one?



15. A curve is such that $\frac{dy}{dx} = x(x+6)(x-6)$.

(a) At how many places on the curve is the gradient zero?

(b) For $x \to \infty$ is the gradient positive or is it negative?

(c) For $x \to -\infty$ is the gradient positive or is it negative?

16. Figure 1 below shows a child's feeding bowl and figure 2 shows the same bowl with the shape of the interior shown. An unfortunate ant has found its way into the bowl and is at the bottom, hoping to get out. However the bowl's surface is very slippery so the ant may not be successful.

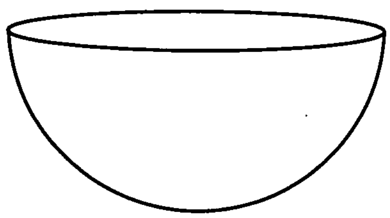


Figure 1

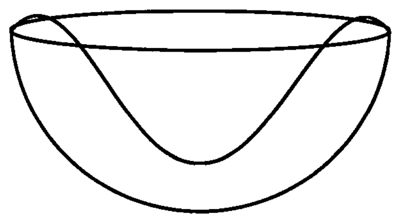


Figure 2

Figure 3 on the right shows that the route the ant must follow, from the bottom of the bowl to the top, can be accurately modelled by part of the curve

$$y = \frac{6x^2}{25} - \frac{2x^3}{125}.$$

The ant starts his (her?) climb to the top but, due to the slippery surface, will slip when the gradient of the slope is $\frac{144}{125}$.

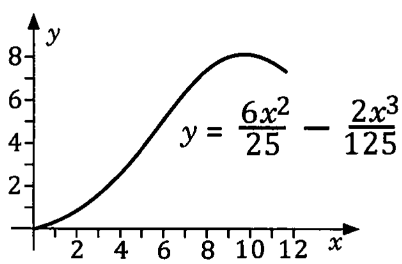


Figure 3

Clearly showing the use of calculus and algebra show that this gradient occurs twice in the section of the curve shown in figure 3, stating the x-coordinate of each of these points and the y-coordinate of the <u>lower</u> point.

Check the x-coordinates just determined using a calculator that has the ability to differentiate functions.